Physics in Medicine & Biology

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To cite this article: A Brahme *et al* 1982 *Phys. Med. Biol.* **27** 1221

Phys. Med. Biol., 1982, Vol. 27, No. 10, 1221–1229. Printed in Great Britain

Solution of an integral equation encountered in rotation therapy

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Received 30 March 1981, in final form 4 December 1981

Abstract. An integral equation relating the lateral absorbed dose profile of a photon beam to the resultant absorbed dose distribution during single-turn rotating-beam therapy has been set up and solved for the case of a cylindrical phantom with the axis of rotation coinciding with the axis of symmetry of the cylinder. In the first approximation the results obtained are also valid when the axis of rotation is somewhat off-centred, even in a phantom that deviates from circular symmetry, provided the rotation is performed in both clockwise and counter clockwise directions. The calculated dose profiles indicate that improved dose uniformity can be achieved using a new type of non-linear wedge-shaped filter, which can easily be designed using the derived general analytic solution to the integral equation.

1. Introduction

In radiation therapy of well defined deep seated tumour volumes rotation therapy is often a treatment method of interest when very high absorbed doses are needed relative to the tolerance level of the surrounding healthy tissues (Johns 1958, Kligerman *et al* 1958, Wichmann and Heinze 1959, Takahashi 1965). This is the case in the treatment of small localised brain tumours, when the absorbed dose to the rest of the brain should be kept low. Other target volumes, where this type of irradiation could be of interest, are the cervix and the bladder with the colon as possible organ at risk. The reverse problem is also of interest, namely when the organ at risk is surrounded by the target volume. This is the case with the medulla when irradiating the surrounding lymphnodes in the head and neck region (cf., Trump *et al* 1961, Proimos 1965, Lax and Brahme 1982).

For all these cases the question may be asked: Which is the desired lateral dose profile in the incident beam that produces a desired radial absorbed dose distribution in the body after one complete rotation? In most cases a uniform absorbed dose to the target volume will be preferred with minimal absorbed dose everywhere else, but other radial dose profiles may also be of interest (Proimos 1979). We will give an exact answer to this question with the assumption of exponential photon absorption.

In order to simplify the mathematical treatment the body cross-section will be assumed to be circularly cylindrical with the axis of rotation coinciding with the cylinder axis. This is obviously a simplification as the body cross-section is rarely circular nor is the target volume placed symmetrically in the body. However, as parallel opposed photon beams generally generate a very uniform dose distribution over the body cross-section this simplification is justified, at least as a first approximation, when two rotations are performed, one in the clockwise and the other in the counter clockwise direction.

2. Mathematical formulation

The depth–dose distribution in a photon beam can, with great accuracy, be approximated by a simple exponential expression characterised by the practical attenuation coefficient, μ_p (cf., Brahme and Svensson 1979; for simplicity we disregard the dose build-up near the surface in the present treatment). Based on this approximation and the assumption of a cylindrical body cross-section with the axis of rotation coinciding with the cylinder axis the dose distribution in the cylinder could be written:

$$d(x, z) = d(x) \exp(-\mu_p z) \tag{1}$$

where $d(x)$ is the dose variation along the positive x-axis ($d(x) \equiv 0$ for $x \leqslant 0$) and z is the distance from the x-axis in the direction of the beam (see figure 1).

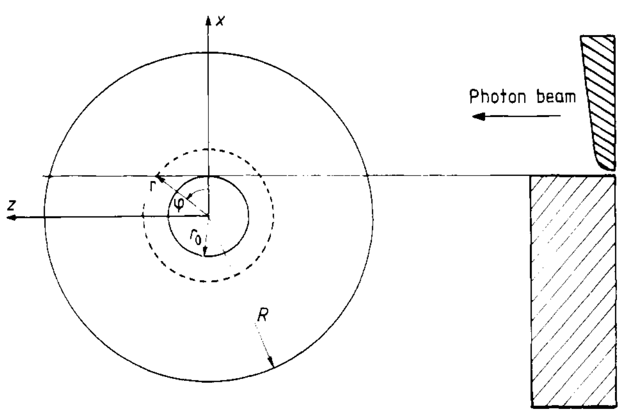

Figure 1. The irradiation geometry and coordinate system used in the calculations. The origin of the rectangular and polar coordinate systems is located at the isocentre of the therapy machine. The location of the beam block and the non-linear wedge-shaped filter is also indicated.

When the beam is now rotated one complete turn, all points at a radial distance r in the cylinder, will receive an absorbed dose contribution equal to the line integral over the absorbed dose distribution along a circle of radius r in the cylinder when irradiated by the stationary beam profile $d(x)$ (assuming the absorbed dose per degree of rotation to be constant). The radial dose variation after one full turn thus becomes:

$$D(r) = \oint_r d(x) \exp(-\mu_p z)\, \mathrm{d}\varphi / \pi. \tag{2}$$

By using the polar coordinate transforms, z and φ can be replaced by r and x since $r^2 = x^2 + z^2$ and $x = r \cos \varphi$. The resulting integral over x becomes:

$$D(r) = \int_0^r \frac{d(x) 2 \cosh \mu_p[(r^2 - x^2)^{1/2}]}{\pi (r^2 - x^2)^{1/2}}\, \mathrm{d}x. \tag{3}$$

The lateral dose distribution in the incident beam $d(x)$ should thus satisfy the above integral equation, where $D(r)$ is the desired dose distribution using rotation therapy. Of special interest for radiation therapy is, of course, the case where $D(r)$ has a constant value independent of r in some radial interval giving a uniform absorbed dose to the tumour volume.

3. General solution of the integral equation

We will first derive the general solution for the case, where $d(x)$ is zero inside a circular region of radius r_0. The integral equation is then:

$$D(r)=\frac{2}{\pi}\int_{r_0}^{r}\frac{d(x)\cosh\left[\mu_p(r^2-x^2)^{1/2}\right]}{(r^2-x^2)^{1/2}}\,dx. \tag{4}$$

Here $0<r_0\leqslant r\leqslant\infty$ and $\mu_p>0$ is a 'small' constant, $D(r)$ is a given function, which is zero for $0\leqslant r<r_0$, and $d(x)$ is the function that we wish to determine in terms of $D(r)$. For simplicity, we suppose that r can take any value up to ∞, although it is possible, and more reasonable, to restrict oneself to a finite interval.

3.1. Heuristic reasoning

If μ_p is very small, corresponding to very high photon energies, the integral equation may be approximated by:

$$D(r)=\frac{2}{\pi}\int_{r_0}^{r}\frac{d(x)}{(r^2-x^2)^{1/2}}\,dx \tag{5}$$

and if r is close to r_0, this equation may be approximated further by:

$$D(r)=\frac{2}{(2r_0)^{1/2}\cdot\pi}\int_{r_0}^{r}\frac{d(x)}{(r-x)^{1/2}}\,dx \tag{6}$$

which is the well-known integral equation of Abel (1823). Equation (6) is known to have the solution (under reasonable regularity conditions on $D(r)$):

$$d(x)=\left(\frac{r_0}{2}\right)^{1/2}\frac{d}{dx}\int_{r_0}^{x}\frac{D(r)}{(x-r)^{1/2}}\,dr. \tag{7}$$

In particular, if $D(r)$ is constant $=D$ for $r\geqslant r_0$, it follows that

$$d(x)=\frac{D(r_0/2)^{1/2}}{(x-r_0)^{1/2}} \tag{8}$$

which has a square root singularity at r_0.

3.2. Explicit solution

We will now treat the general case when $\mu_p\neq 0$ and r is not necessarily close to r_0. It is then reasonable to expect a similar, although slightly more complicated, solution to the general equations (3) and (4). The Abel integral equation (6) is usually solved by noting that it is a convolution equation and by using Laplace transforms. Equation (4) is not a convolution equation; it is, however, a so-called generalised Abel integral equation, to which well known methods can be applied (cf., Handbuch der Physik 1956). However, by a simple transformation of variables equation (4) can be transformed to a convolution equation simply by replacing the old variables (x, r) by a new set of variables (t, y) given by the formulae

$$x^2 = t + r_0^2; \qquad r^2 = y + r_0^2. \tag{9}$$

Equation (4) is transformed into

$$D[(y+r_0^2)^{1/2}] = \frac{1}{\pi}\int_0^y \frac{d[(t+r_0^2)^{1/2}]}{(t+r_0^2)^{1/2}} \cdot \frac{\cosh[\mu_p(y-t)^{1/2}]}{(y-t)^{1/2}}\,\mathrm{d}t \tag{10}$$

which is a convolution equation to which we will now apply the Laplace transform.

Let us first rewrite equation (10) as:

$$g(y) = \int_0^y e(t) f(y-t)\,\mathrm{d}t \tag{11}$$

where

$$e(t) = \frac{d[(t+r_0^2)^{1/2}]}{\pi(t+r_0^2)^{1/2}}$$

$$g(y) = D[(y+r_0^2)^{1/2}]$$

and

$$f(y-t) = \frac{\cosh[\mu_p(y-t)^{1/2}]}{(y-t)^{1/2}}.$$

The Laplace transform H of a function h defined on $(0, \infty)$ is defined by (we suppose that the integral converges at least for some s)

$$H(s) = \int_0^\infty \mathrm{e}^{-sy} h(y)\,\mathrm{d}y \tag{12}$$

where s can take complex values (cf., Handbuch der Physik 1956).

Using Laplace transforms, the convolution equation (11) is transformed into

$$G(s) = E(s) \cdot F(s)$$

so that

$$E(s) = G(s)/F(s). \tag{13}$$

$G(s)$ is the Laplace transform of the given function $g(y) = D[(y+r_0^2)^{1/2}]$ and $E(s)$ is the Laplace transform of the uniquely determined function that we are looking for. The Laplace transform $F(s)$ of $f(y)$ can be obtained by direct integration according to equation (12):

$$F(s) = \int_0^\infty \frac{\cosh \mu_p \sqrt{y}}{\sqrt{y}}\,\mathrm{e}^{-sy}\,\mathrm{d}y$$

which, after the variable transform $y = \xi^2$, becomes

$$F(s) = \int_{-\infty}^{\infty} \exp(\mu_p \xi - s\xi^2)\,\mathrm{d}\xi = \sqrt{\frac{\pi}{s}}\exp(\mu_p^2/4s). \tag{14}$$

Thus equation (13) is reduced to

$$E(s)=\sqrt{\frac{s}{\pi}}\exp(-\mu_{\mathrm{p}}^{2}/4s)G(s). \tag{15}$$

3.2.1. D is constant. We now wish to determine $e(t)$ and start with the particular case when $D(r)=\text{constant}=D$. In this case we have $G(s)=D/s$ so that

$$E(s)=\frac{1}{\sqrt{s\pi}}\exp(-\mu_{\mathrm{p}}^{2}/4s)\cdot D. \tag{16}$$

The explicit function $e(t)$ can now be found by using the inverse Laplace transform. Using the table for inverse Laplace transforms in Erdélyi (1954), we obtain

$$e(t)=\frac{D}{\pi}\frac{1}{\sqrt{t}}\cos\mu_{\mathrm{p}}\sqrt{t}. \tag{17}$$

After introduction of the old variables x and r instead of t and y equation (17) takes the form

$$d(x)=D\cdot\frac{x}{(x^{2}-r_{0}^{2})^{1/2}}\cdot\cos[\mu_{p}(x^{2}-r_{0}^{2})^{1/2}] \tag{18}$$

for $x\geqslant r_0$ and $d(x)=0$ for $x<r_0$.

3.2.2. D is not a constant. We now consider the more general case, when D is not constant. We assume for simplicity that $D(r)$ is a piecewise continuously differentiable function. For technical reasons, we rewrite equation (15) as

$$E(s)=\frac{1}{\sqrt{s\pi}}\cdot\exp(-\mu_{\mathrm{p}}^{2}/4s)\cdot sG(s). \tag{19}$$

Using well known properties of the Laplace transform, this implies that for reasonable $g:s$

$$e(y)=\frac{\mathrm{d}}{\mathrm{d}y}\int_{0}^{y}\frac{D}{\pi}\frac{\cos[\mu_{\mathrm{p}}(y-t)^{1/2}]}{(y-t)^{1/2}}g(t)\,\mathrm{d}t \tag{20}$$

i.e., if we go back to the original variables

$$d(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_{r_0}^{x}\frac{\cos[\mu_{\mathrm{p}}(x^{2}-r^{2})^{1/2}]}{(x^{2}-r^{2})^{1/2}}D(r)r\,\mathrm{d}r \tag{21}$$

which is the explicit formula, from which we can calculate $d(x)$, when $D(r)$ is a given piecewise continuously differentiable function.

3.3. Relation to computed tomography

It is very interesting to note that the present problem is related to the general problem of CT scanners; namely, how to obtain the two dimensional distribution of photon mass attenuation properties in a body from a number of one dimensional projections of the photon absorption in different directions. It can be shown that this problem also leads to an integral equation similar to that of Abel (cf. Cormack 1980) and thus also similar to equations (6) and (3) above. The present problem may be regarded as the reverse problem to that of CT scanning as the desired dose distribution is

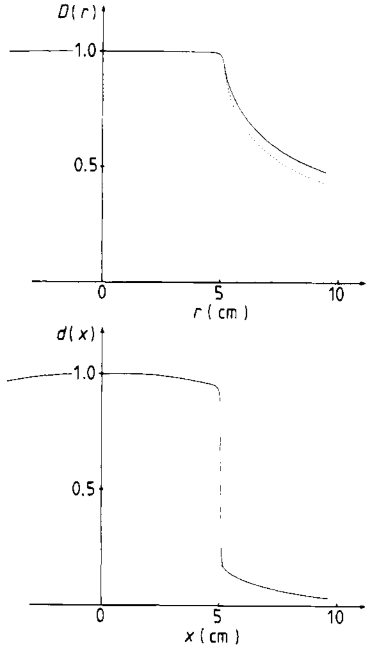

Figure 2. The lateral dose profile $d(x)$ across the diameter of the cylinder when irradiated by the stationary beam is shown in the lower half of the figure for the case of a central target volume. The resulting radial dose distribution in rotation therapy $D(r)$ is shown in the upper half of the figure. In figures 2 and 3 the solid curves include the effects of both primary and scattered photons whereas the dashed curves include only the primary photons. The dose distributions are normalised to unity at the maximum dose values. It is seen that the ideal incident dose profile across the rotation axis should decrease slowly with the radius.

generally known and the incident dose distribution profiles are required instead. However, there exists a major difference as the incident absorbed dose profiles must be larger than, or equal to, zero, whereas the filter functions used in the back-projection of photon absorption profiles in CT scanning also take negative values. This sets strong restrictions on the possibilities of generating a certain type of dose distribution in, for example, rotation therapy, whereas, in principle, any photon attenuation distribution can be reconstructed from measured absorption profiles.

4. Implications for radiation therapy

Two principally different classes of applications of the above formula in radiation therapy can be distinguished. The first is with well defined central target volume where a uniform absorbed dose is wanted with smallest possible dose to surrounding healthy tissues (see figure 2). For such a central cylindrical target volume of radius r, the desired dose variation becomes simply (equation (18) with $r_0=0$)

$$d(x)=D\cos(\mu_p|x|) \qquad |x|\leqslant r \tag{22}$$

and of course $d(x)$ should ideally be zero outside this interval. It is interesting to observe that under the assumption of exponential photon absorption the lateral dose variation should ideally decrease slowly with the distance from the centre according to equation (22). If the photon absorption in the semicylindrical volume in front of

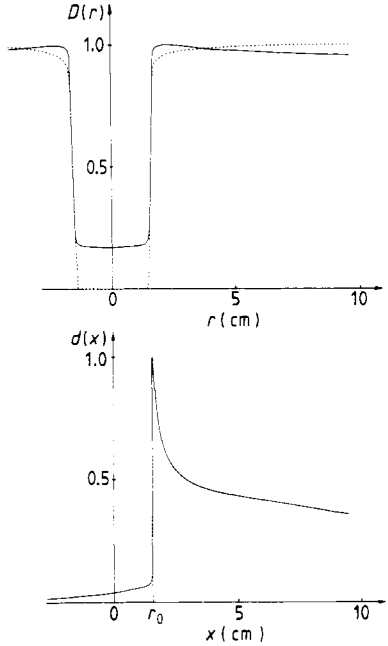

Figure 3. The lower and upper curves respectively, show the lateral dose profile $d(x)$ and the radial dose profile in rotation therapy $D(r)$ for the case of a central organ at risk. By using a cut-off at the singularity, of about three times the plateau level, a fair uniformity over the target volume and a very high dose gradient near the organ at risk are obtained (symbols as in figure 2).

the diameter is taken into account the dose profile in the incident photon beam, $d_{\text{in}}(x)$, should be

$$d_{\text{in}}(x) = D \cos(\mu_{\text{p}}|x|) \exp \mu_{\text{p}}[(R^2 - x^2)^{1/2} - R] \tag{23}$$

which for small x and μ_{p} behaves like $d_{\text{in}}(x) = D(1 - \mu_{\text{p}} x^2(\mu_{\text{p}} + \frac{1}{2}R))$. The resultant primary dose distribution in the cylinder outside r_0 is obtained simply by inserting equation (22) in equation (3) (inside r_0, $D(r) \equiv D$). When the dose contribution from scattered photons is also to be included $d(x)$ can no longer be set to zero outside r_0. The scattered dose contribution to $d(x)$ has been calculated as described by Nilsson and Brahme (1981). The result is shown in figure 2. The solid line curves include the effect of scattered photons whereas this is disregarded in the dashed line curves.

The second application of interest for the present theory is with a cental organ at risk surrounded by the target volume (see figure 3). Most often one would wish $D(r)$ to be constant outside some radius r_0 and zero inside r_0. The resultant dose distribution along the radius of the cylinder is given exactly by equation (18). To get a uniform dose distribution outside r_0 the absorbed dose should theoretically increase to infinity as r approaches r_0. However, in practice a finite value at least a few times larger than the peripheral dose $d(R)$ at some large radius R is sufficient, as shown experimentally by Lax and Brahme (1982). Mathematically this can be understood, as the inverse square root singularity of equation (18) contains a finite integral dose. Thus by limiting the absorbed dose to the theoretical value at some small distance $x - r_0 = \delta$ outside r_0 and extending it the same distance δ inside r_0 the correct integral dose contribution is obtained due to the properties of the square root singularity. Thus by making a non-linear wedge-shaped filter with a transmission profile according to the function

$d(x)$ in figure 3 the target volume can be irradiated uniformly with minimal dose to the organ at risk on the central axis (see figure 1 and Lax and Brahme (1982)).

For large radii and μ_p values (e.g., $r_0^2+9\pi^2/4\mu_p^2>x^2>r_0^2+\pi^2/4\mu_p^2$) equation (18) results in negative doses, which cannot be realised experimentally. In practice this problem is avoided by choosing a high photon energy and consequently a small μ_p for large body cross-sections.

Because of the influence of scattered photons it is physically impossible to make $d(x)=0$ inside r_0 especially as the dose level near r_0 is very high. Therefore, in practice, a compromise must be sought between the height of the peak dose value and the allowable scattered dose level inside r_0. This compromise is principally determined by the ratio of the tolerance level of the organ at risk relative to the desired absorbed dose in the target volume. If this ratio is very small a fairly large value of the parameter δ might be necessary. A typical situation is shown by the solid curves in figure 3, where δ is equal to 0.1 mm and $D(0)$ is about 20% of the dose in the target volume.

In both the above cases, instead of rotating the photon beam alternately in the clockwise and counter-clockwise directions it is advantageous to make the incident dose distribution $d(x)$ symmetric across the rotation axis such that $d(-x)\equiv d(x)$. This procedure is useful for perfectly circular target volumes, but should not be used for other shapes to allow an accurate tangential adjustment of the edge of the field to the target volume (Lax and Brahme 1982).

Acknowledgments

We are indebted to Professor Yngve Domar, Uppsala, for suggesting the variable transform in equation (9).

Résumé

Solution d'une équation intégrale rencontrée en thérapie à champ tournant.

Nous avons établi une équation intégrale reliant le profil latéral de la dose absorbée d'un faisceau de photons à la distribution résultante de la dose absorbée lors d'une thérapie à champ tournant sur un seul tour. Nous avons résolu cette équation dans le cas d'un fantôme cylindrique dont l'axe de rotation coincide avec l'axe de symétrie du cylindre. En première approximation, les résultats obtenus sont également valables quand l'axe de rotation est quelque peu décentré et même pour un fantôme s'écartant de la symétrie circulaire à condition que la rotation ait lieu dans les deux sens (sens des aiguilles d'une montrre et sens inverse). Les profils de dose calculés indiquent que l'on peut obtenir une amélioration de l'uniformité de la dose en utilisant un nouveau type de filtre non linéaire en coin aisément calculable à parti de la solution analytique générale de l'équation intégrale.

Zusammenfassung

Lösung einer Integralgleichung, die in der Rotationstherapie verwendet wird.

Eine Integralgleichung, die das laterale Energiedosisprofil eines Photonenstrahls mit der resultierenden Energiedosisverteilung während einer Rotationsbestrahlung verbindet wunde aufgestellt und für den Fall eines Zylinderphantoms, dessen Symmetrieachse mit der Rotationsachse zusammenfällt, gelöst. In erster Näherung sind die erhaltenen Resultate auch gültig, wenn die Rotationsachse etwas außerhalb des Zentrums liegt, sogar in einem Phantom, das von der Kreissymmetrie abweicht, vorausgesetzt, die Rotation wird sowohl im Uhrzeigersinn wie auch gegen den Uhrzeigersinn durchgeführt. Die berechneten Dosisprofile zeigen, daß eine verbesserte Dosisgleichförmigkeit erreicht werden kann, wenn man einen neuen Typ nicht-linearer keilförmiger Filter vewendet, die mit Hilfe der abgeleiteten allgemeinen analytischen Lösung der Integralgleichung leicht konstruiert weden können.

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